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APPLICATION OF THE OPTIMIZATION METHODS TO SOLUTION OF THE CAUCHY-DIRICHLET PROBLEM FOR POISSON EQUATION

H.F.GULIYEV*, Y.S.GASIMOV*, S.M.ZEYNALLI**
 *Institute of Applied Mathematics, Baku State University
 **Ganja State University, Ganja, Azerbaijan
 h.kuliyev@rambler.ru, ysfgasimov@yahoo.com

In the paper ill-posed Cauchy-Dirichlet problem is considered for the Poisson equation. The problem is reduced to the optimal control problem and regularized. Optimization methods are applied to the solution of the obtained problem.

Key words: Ill-posed problem, Cauchy-Dirichlet problem, optimal control, Poisson equation

1. Introduction

It is known that Cauchy-Dirichlet problem for the elliptic equations, particularly, for the Poisson equation is an ill-posed problem (see [1], [2]). Necessity of investigation of the ill-posed problems of mathematical physics and their well-posed formulation was first noted by A.N.Tikhonov in 1943. It was stimulated by the fact that some physical processes mathematically are described by such problems. Systematical investigation of these problems began in the 50s of the last century, and various methods were applied to their solution.

In [3] optimal control methods are applied to the solution of the ill-posed Cauchy-Dirichlet problem for the Poisson equation. However, the form of the introduced there functional leads to essential complication of the adjoint boundary problem. Here, we consider the same problem and introduce a more natural functional. As a result, the adjoint problem takes a rather simple form. Also, we note and avoid some incorrectness in the formulas obtained in [3].

Note that the interest to the investigation of this problem is encouraged with the strong relations of the Poisson equation with several applications such as, for instance, problems of electrostatics, mathematical engineering, and theoretical physics.

2. Problem Statement

In the domain $Q = \{(x, t) : 0 < x < \pi, -1 < t < 1\}$ the following boundary problem is considered

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = f(x, t), \quad (x, t) \in Q, \quad (1)$$

$$u(0, t) = 0, \quad u(\pi, t) = 0, \quad t \in (-1, 1), \quad (2)$$

$$u(x, -1) = \varphi_0(x), \quad \left. \frac{\partial u(x, t)}{\partial t} \right|_{t=-1} = \varphi_1(x), \quad x \in (0, \pi) \quad (3)$$

It is supposed that $\frac{\partial u(x,1)}{\partial t} \in U_\delta$, where U_δ is a convex closed set from $L_2(0, \pi)$ that includes 0 and

$$f \in L_2(Q), \varphi_0 \in W_2^1(0, \pi), \varphi_1 \in L_2(0, \pi). \quad (4)$$

It is known [4], [5] that this is ill-posed in $L_2(Q)$ problem.

In [3] the following problem is introduced in correspondence to the problem (1)-(3): To minimize on U_δ the functional

$$I(v) = \int_0^\pi \left[\frac{\partial u(x,-1)}{\partial x} - \varphi_0(x) \right]^2 dx \quad (5)$$

subject to

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad (x,t) \in Q, \quad (1)$$

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t \in (-1,1), \quad (2)$$

$$\frac{\partial u(x,t)}{\partial t} \Big|_{t=-1} = \varphi_1(x), \quad \frac{\partial u(x,t)}{\partial t} \Big|_{t=1} = v(x), \quad x \in (0,\pi) \quad (6)$$

where $v \in U_\delta \subseteq L_2(0, \pi)$ is a control function. However, as will be shown below it is natural to consider the following optimal control problem in correspondence to the problem (1)-(3): To minimize on U_δ the functional

$$J(v) = \int_0^\pi [u(x, -1) - \varphi_0(x)]^2 dx, \quad (7)$$

subject to

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad (x,t) \in Q, \quad (1)$$

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t \in (-1,1), \quad (2)$$

$$\frac{\partial u(x,t)}{\partial t} \Big|_{t=-1} = \varphi_1(x), \quad \frac{\partial u(x,t)}{\partial t} \Big|_{t=1} = v(x), \quad x \in (0,\pi). \quad (6)$$

As is known in the optimal control theory (7), (1), (2), (6) is also ill-posed problem [4], [8].

Note that if $f \in L_2(Q)$, $\varphi_1 \in L_2(0, \pi)$, $v \in L_2(0, \pi)$, then the boundary problem (1), (2), (6) has the unique solution from $W_2^1(Q)$ [6].

3. Regularization of the optimal control problem

As is known one of the effective tools to the solution of the ill-posed problems is the regularization method [2], [7]. In our case the functional $\alpha \int_0^\pi |v(x)|^2 dx$, ($\alpha > 0$) may be taken as a stabilizer. Thus instead of (7) we can consider the minimization of the functional

$$J_\alpha(v) = J(v) + \alpha \int_0^\pi |v(x)|^2 dx = \int_0^\pi [u(x, -1) - \varphi_0(x)]^2 dx + \alpha \int_0^\pi |v(x)|^2 dx \quad (8)$$

in the class U_δ within (1), (2), (6).

Let $u(x, t; v)$ be a solution of the problem (1), (2), (6) corresponding to the given control $v \in U_\delta$, and $u(x, t; v)$ -solution of this problem by $v(x) \equiv 0$.

Let's introduce the denotations

$$\alpha(v_1, v_2) = \int_0^\pi [u(x, -1; v_1) - u(x, -1; 0)][u(x, -1; v_2) - u(x, -1; 0)] dx + \alpha \int_0^\pi v_1(x) v_2(x) dx,$$

$$L(v) = \int_0^\pi [\varphi_0(x) - u(x, -1; 0)][u(x, -1; v) - u(x, -1; 0)] dx,$$

where $\alpha(v_1, v_2)$ indeed is a bilinear continuous form on U_δ , $L(v)$ -linear functional on U_δ . Considering these denotations the functional (8) may be rewritten as

$$J_\alpha(v) = \alpha(v, v) - 2L(v) - \int_0^\pi [u(x, 1; 0) - \varphi_0(x)]^2 dx.$$

Since $\alpha(v_1, v_2)$ is a bilinear continuous symmetric form that satisfies to the condition

$$\alpha(v, v) \geq c \|v\|^2, \quad c = \text{const} > 0,$$

considering the well known theorem from [8] the following theorem holds true

Theorem 1. For the optimal control problem (8), (10), (2), (6) there exists the unique element $v \in U_\delta$ such that

$$J_\alpha(\bar{v}) = \inf_{v \in U_\delta} J_\alpha(v).$$

3. The condition of optimality

On the basis of the theorem from [8] may be easily proved the following optimality condition for the problem (8), (1), (2), (6).

Theorem 2. In order to $\bar{v} \in U_\delta$ be an optimal control for the problem (8), (1), (2), (6) it is necessary and sufficient satisfying of the inequality

$$\langle J_{\alpha v}(\bar{v}), v - \bar{v} \rangle \geq 0 \quad \forall v \in U_\delta,$$

i.e.

$$\int_0^\pi [u(x, -1; \bar{v}) - \varphi_0(x)] u_v(x, -1; \bar{v}) [v(x) - \bar{v}(x)] dx + \alpha \int_0^\pi \bar{v}(x) [v(x) - \bar{v}(x)] dx \geq 0 \quad \forall v \in U_\delta, \quad (9)$$

where $J_{\alpha v}$ means Gateaux derivative of the functional J_α with respect to v ; $\langle, \rangle, ;, >$ - scalar product in the space $L_2(0, \pi)$; $u_v(x, t; v)$ is a derivative of the solution of the problem (1), (2), (6) with respect to v .

Let's transform the inequality (9). For this purpose linear boundary problem (1), (2), (6) write in the operator form

$$Au = F \equiv \{f, \varphi_1, v\},$$

with linear operator $A : W_2^1(Q) \rightarrow L_2(Q) \times L_2(0, \pi) \times L_2(0, \pi)$.

Then solution $u(x, t; v)$ of this operator equation may be written as

$$u(x, t; v) = A^{-1}F$$

considering existence of A^{-1} (see [6], [9]). Taking derivative of this solution on the direction $v - \bar{v}$ one may obtain

$$u_v(x, t; \bar{v})[v - \bar{v}] = u(x, t; v) - u(x, t; \bar{v}).$$

Then the inequality (9) takes a form

$$\int_0^\pi [u(x, -1; \bar{v}) - \varphi_0(x)] [u(x, -1; v) - u(x, -1; \bar{v})] dx + \alpha \int_0^\pi \bar{v}(x) [v(x) - \bar{v}(x)] dx \geq 0 \quad \forall v \in U_\delta. \quad (10)$$

Let's introduce the following adjoint boundary problem

$$\frac{\partial^2 \psi}{\partial t^2} + \frac{\partial^2 \psi}{\partial x^2} = 0, \quad (x, t) \in Q, \quad (11)$$

$$\psi(0, t) = \psi(\pi, t) = 0, \quad t \in (-1, 1), \quad (12)$$

$$\left. \frac{\partial \psi(x, t)}{\partial t} \right|_{t=-1} = u(x, -1; \bar{v}) - \varphi_0(x), \quad \left. \frac{\partial \psi(x, 1)}{\partial t} \right|_{t=1} = 0, \quad x \in (0, \pi). \quad (13)$$

Note that the problem (11)-(13) has the unique solution from $W_2^1(Q)$ [6].

Let's transform the first term of the inequality (10) using this boundary problem. If to define $\tilde{u}(x, t) = u(x, t; v) - u(x, t; \bar{v})$, then it is clear that

$$\iint_Q \left[\frac{\partial^2 \tilde{u}}{\partial t^2} + \frac{\partial^2 \tilde{u}}{\partial x^2} \right] \psi(x, t; \bar{v}) dx dt = 0.$$

From this integrating by parts and using the conditions (2), (6), (12), (13) one may get

$$\begin{aligned} \iint_Q \left[\frac{\partial^2 \tilde{u}}{\partial t^2} + \frac{\partial^2 \tilde{u}}{\partial x^2} \right] \psi(x, t; \bar{v}) dx dt &= \int_0^\pi [v(x) - \bar{v}(x)] \psi(x, 1; \bar{v}) dx + \\ &+ \int_0^\pi [u(x, -1; v) - u(x, -1; \bar{v})] \frac{\partial \psi(x, -1; \bar{v})}{\partial t} dx = 0. \end{aligned} \quad (14)$$

Applying (13) from (14) may be obtained

$$\int_0^\pi [u(x, -1; \bar{v}) - \varphi_0(x)] [u(x, -1; v) - u(x, -1; \bar{v})] dx = - \int_0^\pi \psi(x, 1; \bar{v}) [v(x) - \bar{v}(x)] dx. \quad (15)$$

It follows from the relations (10) and (15) that

$$\int_0^\pi [-\psi(x, 1; \bar{v}) + \alpha \bar{v}(x)] [v(x) - \bar{v}(x)] dx \geq 0 \quad \forall v \in U_\delta. \quad (16)$$

Thus we obtain a condition of optimality in the form of the following theorem

Theorem 3. In order to the element $\bar{v}(x) \in U_\delta$ be an optimal control for the problem (8), (1), (2), (6), it is necessary and sufficient that this element satisfy to the boundary problems (1), (2), (6) and (11)-(13) and variational inequality (16).

4. Application of the Fourier method

Let's seek the solutions of the boundary problems (1), (2), (6) and (11)-(13) in the form

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) \chi_k(x), \quad \psi(x, t) = \sum_{k=1}^{\infty} \psi_k(t) \chi_k(x),$$

where

$$\chi_k(x) = \sqrt{\frac{2}{\pi}} \sin kx; \quad \lambda_k = -k^2, \quad k = 1, 2, \dots \quad (17)$$

are the systems of orthonormal eigenfunctions and eigenvalues of the spectral problem
 $\chi''(x) = \lambda\chi(x)$, $\chi(0) = \chi(\pi) = 0$,

and $u_k(t)$, $\psi_k(t)$ - sought functions.

From (1), (2), (6); (11)-(13) and (16) correspondingly is obtained

$$\begin{cases} \ddot{u}_k(t) - k^2 u_k(t) = f_k(t), & t \in (-1, 1), \\ \dot{u}_k(-1) = \varphi_{1k}, \quad \dot{u}_k(1) = \bar{v}_k, & k = 1, 2, \dots, \end{cases} \quad (18)$$

$$\begin{cases} \ddot{\psi}_k(t) - k^2 \psi_k(t) = 0, & t \in (-1, 1) \\ \dot{\psi}_k(-1) = u_k(-1) - \varphi_{0k}, \quad \dot{\psi}_k(1) = 0, & k = 1, 2, \dots, \end{cases} \quad (19)$$

$$[-\psi_k(1) + \alpha \bar{v}_k][v_k - \bar{v}_k] \geq 0, \quad \forall \psi_k, \quad k = 1, 2, \dots \quad (20)$$

Here $f_k(t)$, φ_{0k} , φ_{1k} , \bar{v}_k , v_k $k = 1, 2, \dots$ are Fourier coefficients of the functions $f(x, t)$, $\varphi_0(x)$, $\varphi_1(x)$, $\bar{v}(x)$, $v(x)$, correspondingly over the system (17).

The solutions of the boundary problems (18) and (19) are represented in the form

$$u_k(t) = \bar{v}_k \frac{chk(t+1)}{ksh2k} - \varphi_{1k} \frac{chk(1-t)}{ksh2k} + \int_{-1}^1 G_k(t; \tau) f_k(\tau) d\tau, \quad (21)$$

$$\varphi_k(t) = -\frac{u_k(-1) - \varphi_{0k}}{ksh2k} chk(1-t), \quad (22)$$

where

$$G_k(t, \tau) = \begin{cases} -\frac{chk(1-\tau)chk(1+t)}{ksh2k}, & t \in [-1, \tau], \\ \frac{1}{k} shk(t-\tau) - \frac{chk(1-\tau)chk(1+t)}{ksh2k}, & t \in [\tau, 1] \end{cases}$$

is a Green function for the problem (18).

From (21), (22) and (20) it is easy to get

$$\begin{aligned} u_k(-1) &= \frac{\bar{v}_k}{ksh2k} - \varphi_{1k} \frac{cth2k}{k} + \int_{-1}^1 G_k(-1; \tau) f_k(\tau) d\tau, \\ -\varphi_k(1) &= \frac{u_k(-1) - \varphi_{0k}}{ksh2k}, \\ [u_k(-1) - \varphi_{0k} + \alpha ksh2k \cdot \bar{v}_k][v_k - \bar{v}_k] &\geq 0 \quad \forall v_k. \end{aligned} \quad (23)$$

The condition (23) may be transformed as

$$\left[-\varphi_{1k} \frac{cth2k}{k} + \bar{v}_k \left(\frac{1}{ksh2k} + \alpha ksh2k \right) - \varphi_{0k} + \int_{-1}^1 G_k(-1; \tau) f_k(\tau) d\tau \right] \cdot (v_k - \bar{v}_k) \geq 0 \quad \forall v_k. \quad (23')$$

Now let's consider the case, when $U_\partial = L_2(0, \pi)$. Then from (23') is obtained

$$\bar{v}_k = \beta_{k\alpha}^{-1} \left[\varphi_{0k} + \varphi_{1k} \frac{cth2k}{k} - \int_{-1}^1 G_k(-1; \tau) f_k(\tau) d\tau \right], \quad (24)$$

where

$$\beta_{k\alpha} = \frac{1 + \alpha k^2 sh^2 2k}{ksh2k}, \quad k = 1, 2, \dots$$

Thus, we find optimal values of the Fourier coefficients \bar{v}_k of the function $\bar{v}(x)$. Then by $\alpha \rightarrow 0$ from (21) (24) is obtained

$$u_{k0}(t) = \lim_{\alpha \rightarrow 0} u_k(t) = \varphi_{0k} chk(t+1) + \frac{\varphi_{1k}}{ksh2k} [ch2k chk(t+1) - chk(1-t)] - \quad (25)$$

$$- chk(t+1) \int_{-1}^1 G_k(-1; \tau) f_k(\tau) d\tau + \int_{-1}^1 G_k(t; \tau) f_k(\tau) d\tau,$$

$$\bar{v}_{k0} = \lim_{\alpha \rightarrow 0} \bar{v}_k = \varphi_{0k} ksh2k + \varphi_{1k} ch2k - ksh2k \int_{-1}^1 G_k(-1; \tau) f_k(\tau) d\tau. \quad (26)$$

Moreover, the solutions $u_k(t)$ found by the formula (21) corresponding to the optimal Fourier coefficients \bar{v}_k , $k = 1, 2, \dots$ found by the formula (24) must satisfy to the limit relations $\lim_{\alpha \rightarrow 0} u_k(-1) = \varphi_{0k}$ that really holds and it is consistent with condition $u(x, -1) = \varphi_0(x)$ from (3). Thus the exact solution of the problem (7), (1), (2), (6) is presented by the formula

$$\bar{v}(x) = \sum_{k=1}^{\infty} \frac{2}{\pi} sh2k \left[k\varphi_{0k} + \varphi_{1k} ch2k - k \int_{-1}^1 G_k(-1; \tau) f_k(\tau) d\tau \right] \sin kx. \quad (27)$$

And the solution of the initial problem (1)-(3) indeed has a form

$$u(x, t) = \sum_{k=1}^{\infty} \frac{2}{\pi} \left\{ \varphi_{0k} chk(t+1) + \frac{\varphi_{1k}}{ksh2k} [ch2k \cdot chk(t+1) - chk(1-t)] - \right. \\ \left. - chk(t+1) \int_{-1}^1 G_k(-1; \tau) f_k(\tau) d\tau + \int_{-1}^1 G_k(t; \tau) f_k(\tau) d\tau \right\} \sin kx.$$

Note. It should be noted that the Green function constructed in [3] is not correct. It may be easily verified by checking out of the jump condition of this function at the point $t = \tau$. Consequently, the formula for the exact solution optimization problem and general solution of the initial problem is not correct.

5. Conclusion

In spite of the main formulas obtained in [3] need to be corrected, the conclusion given there generally is true. Really, the Fourier coefficients of the function $\bar{v}(x)$ and consequently of the solution of the initial problem $u(x, t)$ will infinitely increase if they will not be compensated by more rapidly decreasing absolute values of the Fourier coefficients φ_{0k} , φ_{1k} and L_2 norm of f_k . The investigations discover the nature of ill-posedness of the Cauchy-Dirichlet problem and regularization method allows one to solve it approximately. It is well demonstrated by the example of Hadamar.

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PUASSON TƏNLİYİ ÜÇÜN QOYULMUŞ KOŞI-DİRİXLƏ MƏSƏLƏSİNİN HƏLLİNƏ OPTİMALLAŞDIRMA ÜSULLARININ TƏTBİQİ

H.F.QULİYEV, Y.S.QASIMOV, S.M.ZEYNALLI

XÜLASƏ

İşdə Puasson tənliyi üçün qoyulmuş qeyri-korrekt Koşi-Dirixle məsələsinə baxılır. Məsələ optimal idarəetmə məsələsinə gətirilir və rəqulyarlaşdırılır. Alınmış məsələnin həllinə optimallaşdırma üsulları tətbiq olunur.

Açar sözlər: qeyri-korrekt məsələ, Koşi-Dirixle məsələsi, optimal idarəetmə, Puasson tənliyi.

ПРИМЕНЕНИЕ МЕТОДОВ ОПТИМИЗАЦИИ К РЕШЕНИЮ ЗАДАЧИ КОШИ-ДИРИХЛЕ ДЛЯ УРАВНЕНИЯ ПУАССОНА

Г.Ф.ГУЛИЕВ, Ю.С.ГАСЫМОВ, С.М.ЗЕЙНАЛЛЫ

РЕЗЮМЕ

В работе рассматривается некорректная задача Коши-Дирихле для уравнения Пуассона. Задача приводится к задаче оптимального управления и регуляризуется. Для решения полученной задачи применяются методы оптимизации.

Ключевые слова: некорректная задача, задача Коши-Дирихле, оптимальное управление, уравнение Пуассона.

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